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LETTER TO THE EDITOR

Verma module of the quantum group $GL(n)_q$ and its q-boson and Heisenberg–Weyl relation realizations

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Abstract. The structure and Verma module of the quantum matrix element algebra $A(n)_q$ of the quantum group $GL(n)_q$ are studied. The method used here is similar to that for studying the structure and Verma module of semisimple Lie algebras. The concept of the Cartan subalgebra, raising and lowering operators and their pairs is defined. The q-boson and the corresponding Heisenberg-Weyl relation realizations are generally studied and the cyclic representations of $A(n)_q$ are obtained. The explicit examples $A(2)_q$ and $A(3)_q$ are discussed in detail.

Quantum groups and quantum algebras are deeply rooted in many integrable nonlinear physical models through the quantum Yang–Baxter equation [1]. The quantum (universal enveloping) algebras [2] and their representation theory have been extensively studied both in the generic case [3] and in the non-generic case [4].

Quantum groups [5] can be specified using the quantum R-matrix that satisfy the quantum Yang–Baxter equation [6]. Floratos studied the representations of $A(2)_q$ of $GL(2)_q$ in terms of the Heisenberg-Weyl relation (HWR) [7]. Weyers proved that $A(n)_q$ can be recast as the HWR provided some invertibility conditions are met [8] and Chakrabarti and Jagannathan developed a procedure constructing these realizations of $A(n)_q$ [9]. For the case $A(2)_q$ with spectrum parameters, its representations have been well studied in the quantum inverse scattering method [6]. A classification of finite-dimensional irreducible representations of $A(2)_q$ can be found in [10].

In this letter we develop a new procedure for studying the structure and Verma module of $A(n)_q$ of $GL(n)_q$. The method used here is similar to that for studying the structure and Verma module of semisimple Lie algebras. The concept of the Cartan subalgebra, raising and lowering matrix elements and their pairs is defined. The q-boson and HWR (Z_n operator) realizations are generally studied using the procedure formulated in [11]. Explicit examples $A(2)_q$ and $A(3)_q$ are discussed in detail.

In this letter Z^+ denotes the set of all non-negative integers, C denotes the complex number field and $C^* = C\backslash\{0\}$.

The quantum group $GL(n)_q$ is a set of $n\times n$ matrices $M=(m_{ij})$, $1\leqslant i,j\leqslant n$, whose matrix elements are non-commuting and satisfy the following bilinear product relations:

$$m_{ij}m_{ik} = q^{-1}m_{ik}m_{ij} \qquad j<k \tag{1a}$$

$$m_{ij}m_{kj} = q^{-1}m_{kj}m_{ij} \qquad i<k \tag{1b}$$

$$m_{ij}m_{kl} = m_{kl}m_{ij} \qquad i<k \text{ and } j>l \tag{1c}$$

$$m_{ij}m_{kl} = m_{kl}m_{ij} + (q^{-1}-q)m_{il}m_{kj} \qquad i<k \text{ and } j<l. \tag{1d}$$

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We also require that the quantum determinant $D_q(M)$ of matrix M defined by

$$D_q(M) = \sum_{s\in S_n} (-q)^{-l(s)} m_{1s_1} m_{2s_2} \ldots m_{ns_n} \tag{2}$$

is not vanishing (where S_n is the symmetric group and $l(s)$ is the minimal number of permutations in s). The quantum determinant $D_q(M)$ has the property that it commutes with all the matrix elements m_{ij}

$$D_q(M) m_{ij} = m_{ij} D_q(M) \tag{3}$$

and thus $D_{ij}(M)$ is a central element [6]. If q is the primitive pth root of unity, we also have:

Proposition 1. If $q^p = 1$, then m_{ij}^p commutes with all the elements m_{kl} of M.

This proposition can be easily proved using equations ($1a$–c) and the following relations

$$m_{ij} m_{kl}^t = m_{kl}^t m_{ij} - q(1-q^{-2t}) m_{kl}^{t-1} m_{il} m_{kj} \qquad \text{for } i<k \text{ and } j<l,\ t\in Z^+. \tag{4}$$

In this letter we would like to study the representations of the matrix elements m_{ij} of M. For this end we define an associative algebra $A(n)_q$ over C with generators m_{ij} and the defining relations (1). We call $A(n)_q$ the quantum matrix element algebra.

We start from the following proposition:

Proposition 2. The set of all antidiagonal matrix elements $\{m_{in+1-i} | 1 \leqslant i \leqslant n\}$ is a maximal set of mutually commuting matrix elements.

This proposition is obvious from the relations ($1c$). Following the terminology of semisimple Lie algebras, we define the subalgebra $H(n)_q$ of $A(n)_q$ generated by $\{m_{in+1-i} | 1 \leqslant i \leqslant n\}$, the Cartan subalgebra.

From proposition 2 it follows that there exists a common eigenvector v_0 of Cartan subalgebra $H(n)_q$ on the algebraic closed field C such that

$$m_{in+1-i} v_0 = \lambda_i v_0 \qquad \lambda_i \in C. \tag{5}$$

To define the Verma module, we need to have a maximal vector killed by the so-called raising generators. What are the raising generators? In fact, the raising generators can be naturally defined using the requirement that the quantum determinant $D_q(M)$ be a non-zero constant in the Verma module we shall define. Noting that $D_q(M)$ commutes with all the matrix element, the requirement that $D_q(M)$ be a non-zero constant becomes

$$D_q(M) v_0 = \Gamma v_0 \tag{6}$$

where $\Gamma \in C$. If we require that

$$m_{ij} v_0 = 0 \qquad \text{for } j > n+1-i \tag{7}$$

then condition (6) is satisfied and $\Gamma = -q^{-[n/2]} \lambda_1 \lambda_2 \ldots \lambda_n$, where $[n/2]$ is the integer part of $n/2$. To ensure $\Gamma \neq 0$, we suppose that $\lambda_i \neq 0$ ($i = 1, 2, \ldots, n$) in the following. On the basis of the above discussion, we can define that the matrix elements m_{ij}, $j > n+1-i$, are the raising matrix elements and m_{ij}, $j < n+1-i$, the lowering ones. In the sense of equations (5) and (7) we call v_0 the maximal vectors.

Recall that in the theory of semisimple Lie algebras, for every positive root α there are a raising generator x_α and a lowering generator y_α such that $[x_\alpha, y_\alpha] = h_\alpha \in \mathrm{H}$ (H is the Cartan subalgebra of the semisimple Lie algebra). Is there a corresponding concept in $\mathrm{A}(n)_q$? Now let us develop such a concept.

Definition 1. A set $\{m_{ij}, m_{kl}\}$ of two matrix elements is said to be a pair if $i+l=j+k=n+1$.

Then we have

Proposition 3. (*a*) For any lowering matrix element m_{ij} there is a unique raising matrix element m_{kl} such that $\{m_{ij}, m_{kl}\}$ is a pair. (*b*) If $\{m_{ij}, m_{kl}\}$ $(i<k)$ is a pair, then $[m_{ij}, m_{kl}] = (q^{-1}-q)m_{il}m_{kj} = (q^{-1}-q)m_{i\,n+1-i}m_{k\,n+1-k} \in H(n)_q$. (*c*) For $A(n)_q$ there are $n(n-1)/2$ pairs.

The proof of this proposition is easy and we omit it here.

After the above preparation we turn to the Verma module of $\mathrm{A}(n)_q$. The Verma module $V(\lambda_i)$ is defined as

$$V(\lambda_i) = \mathrm{A}(n)_q v_0. \tag{8}$$

Then the following proposition is obvious:

Proposition 4. The Verma module $V(\lambda_i)$ of $A(n)_q$ is spanned by

$$\{X(k_{ij}) = \Pi'_{ij} m_{ij}^{k_{ij}} v_0 \mid j<n+1-i,\ k_{ij} \in Z^+\} \tag{9}$$

where the symbol $'$ means that the product is the ordered product.

It is worth noting that the Cartan matrix elements are not always diagonal on the vectors $X(k_{ij})$ (see case $\mathrm{A}(3)_q$ below). Now we do not have a general principle to choose a linearly independent basis for $V(\lambda_i)$, on which all the Cartan matrix elements are diagonal. However, such a base for $\mathrm{A}(2)_q$ and $\mathrm{A}(3)_q$ is worked out in this letter (see below).

If $q^p=1$, there exists a submodule $I(\mu_{ij})$ of $V(\lambda_i)$ generated by $\{m_{ij}^p - \mu_{ij} \mid j<n+1-i, \mu_{ij} \in C\}$. Then we can define a quotient module $W(\lambda_i, \mu_{ij}) = V(\lambda_i)/I(\mu_{ij})$, which is a finite-dimensional module and is spanned by

$$\{X(k_{ij}) = X(k_{ij}) \text{ Mod } I(\mu_{ij}) \mid j<n+1-i, 0 \leqslant k \leqslant p-1\}. \tag{10}$$

The representation induced on $W(\lambda_i, \mu_{ij})$ has the cyclic property $m_{ij}^p = \mu_{ij}$ $(j<n+1-i)$. However, it is not a pure cyclic representation because of $m_{ij}^p = 0$ $(j>n+1-i)$. In order to obtain a pure cyclic representation we can use the q-boson method formulated in [11]. We will describe the method below.

Now we turn to the explicit examples $\mathrm{A}(2)_q$ and $\mathrm{A}(3)_q$.

For the case $\mathrm{A}(2)_q$ the Verma module is spanned by

$$\{X(k) = m_{11}^k v_0 \mid k \in Z^+, m_{12}v_0 = \lambda_1 v_0, m_{21}v_0 = \lambda_2 v_0, m_{22}v_0 = 0\}. \tag{11}$$

We prove that $X(k)$, $k \in Z^+$, form a basis for $V(\lambda_1, \lambda_2)$ in the case of $q^p \neq 1$. Using the basic commutation relations we have

$$\begin{aligned} m_{12}X(k) &= \lambda_1 q^k X(k) \\ m_{21}X(k) &= \lambda_2 q^k X(k) \\ m_{22}X(k) &= -q^{-1}q^k(q^{-k}-q^k)\lambda_1\lambda_2 X(k-1) \\ m_{11}X(k) &= X(k+1). \end{aligned} \tag{12}$$

Noting that $X(k)$ are eigenvectors of different eigenvalues $\lambda_1 q^k$ of m_{12} in the case $q^p \neq 1$, we get our conclusion. We can also prove that in the case $q^p \neq 1$ the representation (12) is an infinite-dimensional irreducible representation.

If $q^p = 1$, we get a submodule $I(\mu)$ generated by $\{m_{11}^p - \mu \mid \mu \in C\}$. Then we obtain a p-dimensional quotient module $W(\lambda, \mu) = V(\lambda)/I(\mu)$ with the following basis

$$\{X(k) = X(k) \text{ Mod } I(\lambda) \mid 0 \leqslant k \leqslant p-1\}. \tag{13}$$

Then the representation induced on $W(\lambda, \mu)$ is the equation (12) with $0 \leqslant k \leqslant p-1$ and the following additional relations

$$m_{11} X(p-1) = \mu X(0). \tag{14}$$

This is an irreducible representation with cyclic conditions $m_{11}^p = \mu$ and $m_{22}^p = 0$. The pure cyclic representation in which $m_{22}^p \neq 0$ will be obtained below in terms of its q-boson realization.

For the case $\mathrm{A}(3)_q$ the Cartan matrix elements are not always diagonal on the vector $X(m, n, r) = m_{12}^m m_{21}^n m_{11}^r v_0$. For example,

$$m_{22} X(m, n, r) = q^{m+n} \lambda_2 X(m, n, r) - (1 - q^{2r}) q^{m+n-2r-1} X(m+1, n+1, r). \tag{15}$$

So we choose a new set of vectors in $V(\lambda_i)$

$$\{Y(m, n, r) = m_{12}^m m_{21}^n \Delta^r \mid \Delta = m_{11} m_{22} - q^{-1} m_{12} m_{21},\ m, n, r \in Z^+\}. \tag{16}$$

Then using the following induction relation

$$X(m, n, r) = \lambda_2^{-1} \Delta X(m, n, r-1) - \lambda_2^{-1} q^{-2r+1} X(m+1, n+1, r-1) \tag{17}$$

we know that $Y(m, n, r)$ is complete. We can also prove that, if $q^p \neq 1$, $Y(m, n, r)$ are linearly independent because they are the eigenvectors of different eigenvalues of the operator $m_{13} + m_{22} + m_{31}$:

$$(m_{13} + m_{22} + m_{31}) Y(m, n, r) = (\lambda_1 q^{m+r} + \lambda_2 q^{m+n} + \lambda_3 q^{n+r}) Y(m, n, r). \tag{18}$$

Therefor $Y(m, n, r)$ from a basis for $V(\lambda_i)$ in the case of $q^p \neq 1$.

The representation on $V(\lambda_i)$ is obtained as

$$\begin{aligned}
& m_{22} Y(m, n, r) = q^{m+n} \lambda_2 Y(m, n, r) \\
& m_{13} Y(m, n, r) = q^{m+r} \lambda_1 Y(m, n, r) \qquad m_{31} Y(m, n, r) = q^{n+r} \lambda_3 Y(m, n, r) \\
& m_{12} Y(m, n, r) = Y(n+1, n, r) \qquad m_{21} Y(m, n, r) = Y(m, n+1, r) \\
& m_{11} Y(m, n, r) \\
& \qquad = q^{-(m+n)} \lambda_2^{-1} Y(m, n, r+1) \\
& \qquad\quad + q^{-(m+n+1)} \lambda_2^{-1} Y(m+1, n+1, r) \\
& m_{23} Y(m, n, r) = -q^{n+r-1} \lambda_1 \lambda_2 (1 - q^{2m}) Y(m-1, n, r) \\
& m_{32} Y(m, n, r) = -q^{m+r-1} \lambda_2 \lambda_3 (1 - q^{2n}) Y(m, n-1, r) \\
& m_{33} Y(m, n, r) \\
& \qquad = q^{-3} \lambda_1 \lambda_2 \lambda_3 (1 - q^{2r}) Y(m, n, r-1) \\
& \qquad\quad + q^{2r-2} \lambda_1 \lambda_2 \lambda_3 (1 - q^{2m})(1 - q^{2n}) Y(m-1, n-1, n-1, r)
\end{aligned} \tag{19}$$

which is an infinite-dimensional irreducible representation if $q^p \neq 1$.

If $q^p=1$, we consider the quotient module $W(\lambda_i, \mu_{ij})$, which is also generated by m_{12}^p, m_{21}^p and Δ^p (in (16) A^p is indeed a central element!). In this case a basis for $W(\lambda_i, \mu_i)$ can be chosen as

$$\begin{aligned}&\{F(m,n,r)=Y(m,n,r) \,\mathrm{Mod}\, I(\mu_i) \mid 0\leqslant m,n,r\leqslant p-1\}\\ &\dim W(\lambda_i,\mu_i)=p^3.\end{aligned} \tag{20}$$

Then the representation on $W(\lambda_i, \mu_i)$ is (19) with $Y(m, n, r)$ replaced by $F(m, n, r)$ $(0\leqslant m, n, r\leqslant p-1)$ and the following relations (letting $\Delta^p=\mu_4$):

$$m_{12}F(p-1,n,r)=\mu_1F(0,n,r) \qquad m_{21}F(m,p-1,r)=\mu_2F(m,0,r)$$

$$\begin{aligned}m_{11}F(p-1,n,r)&\\ &=q^{-(n-1)}\lambda_2^{-1}F(p-1,n,r+1)\\ &\quad+q^{-n}\lambda_2^{-1}\mu_1F(0,n+1,r)\end{aligned}$$

$$\begin{aligned}m_{11}F(m,p-1,r)&\\ &=q^{-(m-1)}\lambda_2^{-1}F(m,p-1,r+1)\\ &\quad+q^{-m}\lambda_2^{-1}\mu_2F(m+1,0,r)\end{aligned}$$

$$\begin{aligned}m_{11}F(m,n,p-1)&\\ &=q^{-(m+n)}\lambda_2^{-1}\mu_4F(m,n,0)\\ &\quad+q^{-(m+n+1)}\lambda_2^{-1}F(m+1,n+1,p-1)\end{aligned}$$

which is also an irreducible representation.

Using the method formulated in [11] we can obtain the q-boson realizations of $\mathrm{A}(n)_q$. The key of the method is to construct a representation of $\mathrm{A}(n)_q$ on the q-Fock space which is isomorphic to the Verma module. For the detailed description of the method, please see [11]. Here we only discuss the cases $\mathrm{A}(2)_q$ and $\mathrm{A}(3)_q$.

Define the q-Fock space $\mathscr{F}_q$ of the q-deformed Heisenberg–Weyl algebra [12] of one q-boson:

$$\mathscr{F}_q:\{|k\rangle=(b^+)^k|0\rangle \mid k\in Z^+,\ b|0\rangle=0,\ Q^{\pm}|0\rangle=|0\rangle\}. \tag{21}$$

The mapping $\phi: V(\lambda)\to\mathscr{F}_q$ defined by ϕ: $X(k)\mapsto|k\rangle$ is a linear space isomorphism. Then we obtain a representation $\Gamma(x)=\phi\rho(x)\phi^{-1}$ of $\mathrm{A}(2)_q$ on $\mathscr{F}_q$, where $x\in\mathrm{A}(2)_q$ and ρ is the Verma representation (12) of $\mathrm{A}(2)_q$. This representation is of the form

$$\begin{aligned}&\Gamma(m_{12})|k\rangle=\lambda_1q^k|k\rangle\\ &\Gamma(m_{21})|k\rangle=\lambda_2q^k|k\rangle\\ &\Gamma(m_{22})|k\rangle=q^{-1}q^k(q^k-q^{-k})|k-1\rangle\\ &\Gamma(m_{11})|k\rangle=|k+1\rangle\end{aligned} \tag{22}$$

from which we immediately obtain the q-boson realization of $\mathrm{A}(2)_q$

$$\begin{aligned}&m_{12}=\lambda_1Q^+ \qquad m_{21}=\lambda_2Q^+\\ &m_{22}=(q-q^{-1})Q^+b \qquad m_{11}=b^+.\end{aligned} \tag{23}$$

By making use of the cyclic representation of q-deformed Heisenberg–Weyl algebra [11]

$$
\begin{aligned}
&b^+ v_k = v_{k+1} && 0 \leqslant k \leqslant p-2 \\
&b^+ v_{p-1} = \xi v_0 && \xi \in C^* \\
&b v_k = [k+\zeta] v_{k-1} && 1 \leqslant k \leqslant p-1 \\
&b v_0 = \xi^{-1}[\zeta] v_{p-1} && \zeta \in C \\
&Q^{\pm} v_k = q^{k \pm \zeta} v_k
\end{aligned}
\tag{24}
$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ we obtain a pure cyclic representation (ζ is non-generic)

$$
\begin{aligned}
&m_{12} v_k = \lambda_1 q^{k+\zeta} v_k \qquad m_{21} v_k = \lambda_2 q^{k+\zeta} v_k \\
&m_{22} v_k = q^{k+\zeta-1}(q - q^{-1})[k+\zeta] v_{k-1} \qquad 1 \leqslant k \leqslant p-1 \\
&m_{22} v_0 = -q^{\zeta-1}\xi^{-1}(q - q^{1})[\zeta] v_{p-1} \\
&m_{11} v_k = v_{k+1} \qquad 0 \leqslant k \leqslant p-2 \\
&m_{11} v_{p-1} = \xi v_0
\end{aligned}
\tag{25}
$$

It is easy to verify that $m_{11}^p = \xi$ and $m_{22}^p = \xi^{-1}\Pi_{k=0}^{p-1}(q^{k+\zeta} - q^{-(k+\zeta)})$.

Using the same method we obtain the q-boson realization of $\mathrm{A}(3)_q$ as

$$
\begin{aligned}
&m_{22} = \lambda_2 Q_1^+ Q_2^+ \qquad m_{13} = \lambda_1 Q_1^+ Q_3^+ \qquad m_{31} = \lambda_3 Q_2^+ Q_3^+ \\
&m_{12} = b_1^+ \qquad m_{21} = b_2^+ \qquad m_{11} = \lambda_2^{-1} Q_1^- Q_2^- (b_3^+ + q b_1^+ b_2^+) \\
&m_{23} = (q - q^{-1})\lambda_1\lambda_2 Q_1^+ Q_2^+ Q_3^+ b_1 \qquad m_{32} = (q - q^{-1})\lambda_2\lambda_3 Q_1^+ Q_2^+ Q_3^+ b_2 \\
&m_{33} = -\lambda_1\lambda_2\lambda_3 q^{-2}(q - q^{-1}) Q_3^+ b_3 + \lambda_1\lambda_2\lambda_3 (q - q^{-1})^2 Q_1^+ Q_2^+ Q_3^{+2} b_1 b_2 .
\end{aligned}
\tag{26}
$$

Then we can immediately obtain its pure cyclic representation.

There is a corresponding relation between the q-boson operators and the HWR (also called Z_n operators: $Z_i X_i = q^{-1} X_i Z_i$)

$$
\begin{aligned}
&b_i^+ \Leftrightarrow Z_i \qquad Q_i^+ \Leftrightarrow X_i \\
&b_i \Leftrightarrow Z_i^{-1}(X_i - X_i^{-1})/(q - q^{-1}).
\end{aligned}
$$

Then we can obtain the HWR realization of $\mathrm{A}(n)_q$ from its q-boson realization. It is obvious that $\mathrm{A}(n)_q$ can be realized in terms of $n(n-1)/2$ q-boson operators or $n(n-1)/2$ HWR. This conclusion is in accord with Weyers' result.

So far we have generally studied the Verma module and its q-boson realization of the quantum matrix element algebra $\mathrm{A}(n)_q$. However, there are some questions which is not clear. How do we choose a basis for $V(\lambda_i)$ on which all the Cartan matrix elements are diagonal? What is the classification of the finite- and infinite-dimensional irreducible representations of $\mathrm{A}(n)_q$. These open problems are under consideration.

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References

[1] Drinfeld V 1985 *Sov. Math. Dokl.* **32** 254
Jimbo M 1987 *Commun. Math. Phys.* **102** 537
[2] Drinfeld V 1986 *Proc. ICM (Berkely, CA)* p 793
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 **11** 247
[3] Rosso M 1988 *Commun. Math. Phys.* **117** 581; 1989 *Commun. Math. Phys.* **124** 307
[4] Lusztig G 1984 *Contemp. Math.* **82** 59
De Concini C and Kac V G 1990 Representations of quantum groups at q roots of 1 *Preprint*
[5] Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613; 1987 *Publ. RIMS* **23** 117
Manin Yu I *Quantum Groups and Non-commutative Geometry*, Centre des Recherches Mathematiques, Montreal University report
[6] Takhtajan L A 1989 Lectures on quantum groups *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory* ed M L Ge and B H Zhao (Singapore: World Scientific)
[7] Floratos E G 1990 *Phys. Lett.* **233B** 395
[8] Weyers J 1990 *Phys. Lett.* **240B** 396
[9] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** 1709
[10] Ge M L, Sun C P and Liu X F 1991 *Preprint* NKIM
[11] Fu H C and Ge M L 1991 The q-boson realization of parametrized cyclic representations of quantum algebras at $q^p = 1$ *J. Math. Phys.* in press
Fu H C 1991 The construction of q-boson realizations of quantum algebras *Commun. Theor. Phys.* in press
[12] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581